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FORMULAE FOR THE ASYMPTOTIC DISTRIBUTION OF HOTELLING'S TRACE UNDER VIOLATIONS

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An asymptotic formula is derived for the nonnull distribution of Hotelling's trace statistic for testing the hypothesis of the equality of the covariance matrices of two multivariate normal populations. The study is further extended to the case where one of the sample covariance matrix is distributed as non-central Wishart and the other sample covariance matrix is distributed as central Wishart.

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PREFACE

This is the final report of research in multivariate distributions supported by Contract F33615-72-C-1400 of the Aerospace Research Laboratories, Air Force Systems Command, United States Air Force. The work reported herein of K. C. S. Pillai was wholly and of Nashat B. Saweris partly accomplished on Project 7071, "Research in Applied Mathematics", and was technically monitored by Dr. P. R. Krishnaiah of the Aerospace Research Laboratories. The work of Nashat B. Saweris was in part supported by David Ross Grant from Purdue Research Foundation.

TABLE OF CONTENTS

SECTION I

ASYMPTOTIC FORMULAE FOR THE DISTRIBUTION OF HOTELLING'S TRACE FOR TESTS OF EQUALITY OF TWO COVARIANCE MATRICES

1. INTRODUCTION

Let $m\,\underset{\sim}{S}_1$ and $n\,\underset{\sim}{S}_2$ be independently distributed $W(m,p,\underset{\sim}{\Sigma}_1)$ and $W(n,p,\underset{\sim}{\Sigma}_2)$, respectively. Chattopadhyay and Pillai [1] have given asymptotic expansions for the c.d.f. and percentiles of $T = m\,\mathrm{Tr}\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ up to terms of order n^{-1} in which the noncentrality was denoted by $(\underset{\sim}{F}) = \mathrm{tr}\,\underset{\sim}{F} = \mathrm{tr}(\underset{\sim}{B}^{-1}\underset{\sim}{A} - \underset{\sim}{I})$,"the deviation matrix", where $\underset{\sim}{B} = \underset{\sim}{\Sigma}_1^{-1}$ and $\underset{\sim}{A} = \underset{\sim}{\Sigma}_2^{-1}$. In their paper, terms involving $f_{ij}f_{kl}$, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$, have been neglected. These terms are taken into consideration in the section, and noncentrality is expressed in the form $(\underset{\sim}{F}s) = \mathrm{tr}\,\underset{\sim}{F}^s$. Table I gives tabulations to show the importance of these terms. Furthermore, Chattopadhyay-Pillai (denoted by C-P hereafter) expansions are extended to terms of order $1/n^2$. It may be noted here that $T = nU^{(P)}$, where $U^{(P)}$ is the statistic studied by Pillai [2] for the test of $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, and the power of this test against alternatives of a one-sided nature was studied by Pillai and Jayachandran [3] for p=2. Recently the exact non-null distribution of Hotelling's trace and tabulations of the power of the same test for small and large deviations of the parameters were studied by Pillai and Sudjana [4] for p=3 and m=4. Some power tabulations are presented in Table I up to terms of order n^{-2}, which show extremely good accuracy compared to the exact values given by Pillai and Sudjana. Some additional tabulations of powers are also presented for p=4.

2. THE METHOD OF ASYMPTOTIC EXPANSION

The notations here as well as in the rest of the section follow those of [1] and [5]. In order to describe the method we will first derive an asymptotic expansion for the percentiles of T using which we will further

obtain that of the c.d.f. of T. It is well known [6] that the statistic $y = m \operatorname{Tr} \underset{\sim}{S}_1 \underset{\sim}{A}$ can be written as $\Sigma_1^p \lambda_j \chi_j^2(m)$, where $\chi_j^2(m)$'s are independent central chi-square variables with m d.f. and λ_j's, $j = 1,\ldots,p$, are the characteristic roots of $\underset{\sim}{U} = \underset{\sim}{A}\,\underset{\sim}{B}^{-1}$.

Let $G(\theta) = \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \leq 2\theta\}$.

Now note that

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{B} \leq 2\theta\} = G_\rho(\theta) = [\Gamma(\rho)]^{-1}\int_0^\theta e^{-t} t^{\rho-1} dt, \tag{1}$$

where $\rho = mp/2$. In $G(\theta)$, as a first approximation, for large n we may replace $\underset{\sim}{A}^{-1}$ by $\underset{\sim}{S}_2$ and consider

$$G(\theta) = \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\,\underset{\sim}{S}_2^{-1} \leq 2\theta\}. \tag{2}$$

Now we may, as suggested in [5], obtain a function $h(\underset{\sim}{S}_2)$ in the elements of $\underset{\sim}{S}_2$ such that

$$G(\theta) = \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)\},$$

and then write $h(\underset{\sim}{S}_2)$ as a series with the first term being a linear function of chi-square variables and successive terms of decreasing order of magnitude. We get

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)\} = \int_R \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2\} \Pr\{d\underset{\sim}{S}_2\}, \tag{3}$$

where $\Pr\{d\underset{\sim}{S}_2\}$ is the probability element of the central Wishart distribution of $\underset{\sim}{S}_2$ and R is the domain of integration of $\underset{\sim}{S}_2$. Now we may expand $\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2\}$ about an origin $(\sigma_{11},\sigma_{22},\ldots,\sigma_{pp},\sigma_{12},\ldots,\sigma_{p-1,p})$ in a Taylor series, where

$$\underset{\sim}{A}^{-1} = (\sigma_{iJ}) \quad i, J = 1,\ldots,p. \tag{4}$$

Thus

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2\}$$

$$=\{\exp[\sum_{i\leq J=1}^{p} (S_{ij} - \sigma_{ij}) \frac{\partial}{\partial\sigma_{ij}}]\} \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \leq 2h(\underset{\sim}{A}^{-1})\}$$

$$= \{\exp[\mathrm{Tr}(\underset{\sim}{S}_2 - \underset{\sim}{A}^{-1})\partial]\}\ \Pr\{m\ \mathrm{Tr}\ \underset{\sim}{S}_1\underset{\sim}{A} \leq 2h(\underset{\sim}{A}^{-1})\}, \tag{5}$$

where

$$\partial(p\mathrm{x}p) = \left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\sigma_{ij}}\right) = \begin{pmatrix} \frac{\partial}{\partial\sigma_{11}} & \frac{1}{2}\frac{\partial}{\partial\sigma_{12}} & \cdots\cdots\cdots & \frac{1}{2}\frac{\partial}{\partial\sigma_{1p}} \\ \frac{1}{2}\frac{\partial}{\partial\sigma_{21}} & \frac{\partial}{\partial\sigma_{22}} & \cdots\cdots\cdots & \frac{1}{2}\frac{\partial}{\partial\sigma_{2p}} \\ \frac{1}{2}\cdot\frac{\partial}{\partial\sigma_{p1}} & \frac{1}{2}\cdot\frac{\partial}{\partial\sigma_{p2}} & \cdots\cdots & \frac{\partial}{\partial\sigma_{pp}} \end{pmatrix} \tag{6}$$

where δ_{ij} is the Kronecker delta. Hence substitution of Eq (5) into Eq (3) and term by term integration for sufficiently large n gives

$$G(\theta) = \int_R \exp[\mathrm{Tr}(\underset{\sim}{S}_2 - \underset{\sim}{A}^{-1})\partial]\ \Pr\{m\ \mathrm{Tr}\ \underset{\sim}{S}_1\underset{\sim}{A} \leq 2h(\underset{\sim}{A}^{-1})\}\ \Pr\{d\underset{\sim}{S}_2\}$$

$$= \theta\ \Pr\{m\ \mathrm{Tr}\ \underset{\sim}{S}_1\underset{\sim}{A} \leq 2h(\underset{\sim}{A}^{-1})\},$$

where

$$\theta = \exp[-\mathrm{Tr}\ \underset{\sim}{A}^{-1}\partial](\Gamma_\rho(n))^{-1}\ |\underset{\sim}{A}|^{n/2}.$$

$$\int_R |\underset{\sim}{S}_2|^{(n-p-1)/2}\ \exp[\mathrm{Tr}(\underset{\sim}{S}_2\partial - (n/2)\underset{\sim}{A}\underset{\sim}{S}_2]d\underset{\sim}{S}_2$$

$$= \exp[-\mathrm{Tr}\ \underset{\sim}{A}^{-1}\partial]\,|\underset{\sim}{I} - (2/n)\ \underset{\sim}{A}^{-1}\ \partial|^{-(n/2)},$$

Now using [7], we get

$$\theta = 1 + \frac{1}{n}\sum\sigma_{rs}\,\sigma_{tu}\,\theta_{st}\,\partial_{ur} + \frac{1}{n^2}\{\frac{4}{3}\sum\sigma_{rs}\sigma_{tu}\sigma_{vw}$$

$$\partial_{st}\partial_{uv}\partial_{wr} + \frac{1}{2}\sum\sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}\} + O(n^{-3}), \tag{7}$$

where Σ denotes the summation over all suffixes r,s,..., each of which ranges from 1 to p. Further, we represent $h(\underset{\sim}{S}_2)$ as

$$h(\underset{\sim}{S}_2) = \theta + h_1(\underset{\sim}{S}_2) + h_2(\underset{\sim}{S}_2) + \ldots\ldots, \tag{8}$$

where $h_s(\underset{\sim}{S}_2)$ is of order n^{-s}. Then Eq (8) may be substituted into $\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2h(\underset{\sim}{A}^{-1})\}$, and by Taylor's expansion we have

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2h(\underset{\sim}{A}^{-1})\}$$

$$= \exp[\{h_1(\underset{\sim}{A}^{-1}) + h_2(\underset{\sim}{A}^{-1}) + \ldots\}D] \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\}$$

$$= [1 + \{h_1(\underset{\sim}{A}^{-1}) + h_2(\underset{\sim}{A}^{-1}) + \ldots\}D + \frac{1}{2}\{h_1(\underset{\sim}{A}^{-1}) +$$

$$h_2(\underset{\sim}{A}^{-1}) + \ldots\}^2 D^2 + \ldots\ldots]. \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\}, \tag{9}$$

where $$D = \frac{\partial}{\partial\theta}. \tag{10}$$

Hence we get

$$G(\theta) = [1 + \frac{1}{n}\Sigma\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + \frac{1}{n^2}\{\frac{4}{3}\Sigma\sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr} +$$

$$\frac{1}{2}\Sigma\sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}\} + 0(n^{-3})][1 + h_1(\underset{\sim}{A}^{-1})D +$$

$$\{h_2(\underset{\sim}{A}^{-1})D + \frac{1}{2}h_1^2(\underset{\sim}{A}^{-1})D^2\} + 0(n^{-3})] \quad \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\}.$$

Now, equating terms of successive order [1], we have

$$\{h_1(\underset{\sim}{A}^{-1})D + \frac{1}{n}\Sigma\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}\} \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\} = 0, \tag{11}$$

$$[h_2(\underset{\sim}{A}^{-1})D + \frac{1}{2}h_1^2(\underset{\sim}{A}^{-1})D^2$$

$$+ \frac{1}{n}\Sigma\sigma_{rs}\sigma_{tu}\{h_1^{(st,ur)}(\underset{\sim}{A}^{-1})D + 2h_1^{(st)}(\underset{\sim}{A}^{-1})\partial_{ur}D$$

$$+ h_1(\underset{\sim}{A}^{-1})\partial_{st}\partial_{ur}D\} + \frac{4}{3n^2}\Sigma\sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr}$$

$$+ \frac{1}{2n^2}\Sigma\sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv}] \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\} = 0, \tag{12}$$

and so on, where $h_1^{(st)}(\underset{\sim}{A}^{-1}) = \partial_{st} h_1(\underset{\sim}{A}^{-1})$ and $h_1^{(st,ur)}(\underset{\sim}{A}^{-1}) = \partial_{ur}\partial_{st}\, h_1(\underset{\sim}{A}^{-1})$. Hence to evaluate $h_1(\underset{\sim}{A}^{-1})$ and $h_2(\underset{\sim}{A}^{-1})$ we have to find

$\partial_{st}\partial_{ur}\ \Pr\{m \operatorname{Tr} \underset{\sim}{S_1}\underset{\sim}{A} \leq 2\theta\}$, $\partial_{st}\partial_{uv}\partial_{wr}\ \Pr\{m \operatorname{Tr} \underset{\sim}{S_1}\underset{\sim}{A} \leq 2\theta\},\ldots$ For this purpose we use perturbation technique [8]. Let

$$J = \Pr\{m \operatorname{Tr} \underset{\sim}{S_1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon})^{-1} \leq 2\theta\},$$

where $\underset{\sim}{\epsilon}(p\mathrm{x}p)$ is a symmetric matrix sufficiently close to $\underset{\sim}{0}(p\mathrm{x}p)$. By Taylor's theorem we get

$$J = \{1 + \Sigma\, \epsilon_{rs}\partial_{rs} + \frac{1}{2!}\Sigma\, \epsilon_{rs}\epsilon_{tu}\partial_{rs}\partial_{tu} + \frac{1}{3!}\Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw} + \frac{1}{4!}\Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\partial_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} + \ldots\ldots\} \Pr\{m \operatorname{Tr} \underset{\sim}{S_1}\underset{\sim}{A} \leq 2\theta\}. \qquad (13)$$

Also by definition we get

$$J = \frac{|\underset{\sim}{B}|^{m/2}}{(2\Pi)^{(mp)/2}} \int_R \exp[-\frac{1}{2}\operatorname{Tr} \underset{\sim}{B}\,\underset{\sim}{Y}\,\underset{\sim}{Y}'] \, d\underset{\sim}{Y},$$

where $m\,\underset{\sim}{S_1} = \underset{\sim}{YY}'$, $\underset{\sim}{Y}(p\mathrm{x}m)$ and $R{:}\{Y{:}m \operatorname{Tr} S_1(\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon})^{-1} \leq 2\theta\}$. Now let $\underset{\sim}{\Gamma}(p\mathrm{x}p)$ be a nonsingular matrix such that

$$\frac{1}{2}\underset{\sim}{\Gamma}'\,\underset{\sim}{B}\,\underset{\sim}{\Gamma} = \underset{\sim}{I}(p\mathrm{x}p) - \underset{\sim}{D}_{\eta},$$

and

$$\frac{1}{2}\underset{\sim}{\Gamma}'\,(\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon})^{-1}\,\underset{\sim}{\Gamma} = \underset{\sim}{I}(p\mathrm{x}p),$$

for $\underset{\sim}{\epsilon}(p\mathrm{x}p)$ sufficiently close to $\underset{\sim}{0}(p\mathrm{x}p)$ and $\underset{\sim}{D}_{\eta} = \operatorname{diag}(\eta_1,\ldots,\eta_p)$. This is possible as $\underset{\sim}{B}$ and $\underset{\sim}{A}^{-1}$ are p.d.

Let

$$\underset{\sim}{Y}(p\mathrm{x}m) = \underset{\sim}{\Gamma}(p\mathrm{x}p)\ \underset{\sim}{Z}(p\mathrm{x}m).$$

Then

$$J = \left(\frac{|\underset{\sim}{I}-\underset{\sim}{D}_\eta E|}{|\underset{\sim}{I}-\underset{\sim}{D}_\eta|}\right)^{-(m|2)} \quad G_\rho(\theta),$$

where $\rho = mp/2$ and E is an operator such that $E\,G_\rho(\theta) = G_{\rho+1}(\theta)$. Now let $E = \Delta + 1$.

Then

$$|\underset{\sim}{I} - \underset{\sim}{D}_\eta E|/|\underset{\sim}{I} - \underset{\sim}{D}_\eta| = |\underset{\sim}{I} - \underset{\sim}{D}_\eta - \underset{\sim}{D}_\eta\Delta|/|\underset{\sim}{I} - \underset{\sim}{D}_\eta|$$

$$= |\underset{\sim}{I} - [\underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon})^{-1} - \underset{\sim}{I}]\Delta|$$

$$= |\underset{\sim}{I} - \underset{\sim}{X}\Delta|, \qquad \text{(say)}.$$

Hence

$$J = |\underset{\sim}{I} - \underset{\sim}{X}\Delta|^{-(m|2)} \quad G_\rho(\theta)$$

$$= \exp[(-m|2)\log|\underset{\sim}{I} - \underset{\sim}{X}\Delta|]\; G_\rho(\theta).$$

Now, if $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{F}$ such that $|ch_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, then for $\underset{\sim}{\epsilon}(p\text{x}p)$ sufficiently close to $\underset{\sim}{0}(p\text{x}p)$ we get $|ch_i(\underset{\sim}{X})| < 1$, $i = 1,\ldots,p$, and

$$J = \exp\{\tfrac{m}{2}\,\mathrm{Tr}\,\underset{\sim}{X}\Delta + \tfrac{m}{4}\,\mathrm{Tr}\,\underset{\sim}{X}^2\Delta^2 + \tfrac{m}{6}\,\mathrm{Tr}\,\underset{\sim}{X}^3\Delta^3 + \tfrac{m}{8}\,\mathrm{Tr}\,\underset{\sim}{X}^4\Delta^4 + \ldots\ldots\}\; G_\rho(\theta)$$

$$\begin{aligned}
&= [1 + \tfrac{m}{2}\,\mathrm{Tr}\,\underset{\sim}{X}\Delta + \{\tfrac{m}{4}\,\mathrm{Tr}\,\underset{\sim}{X}^2 + \tfrac{m^2}{8}(\mathrm{Tr}\,\underset{\sim}{X})^2\}\,\Delta^2 \\
&+ \{\tfrac{m}{6}\,\mathrm{Tr}\,\underset{\sim}{X}^3 + \tfrac{m^2}{8}(\mathrm{Tr}\,\underset{\sim}{X})(\mathrm{Tr}\,\underset{\sim}{X}^2) + \tfrac{m^3}{48}(\mathrm{Tr}\,\underset{\sim}{X})^3\}\,\Delta^3 \\
&+ \{\tfrac{m}{8}\,\mathrm{Tr}\,\underset{\sim}{X}^4 + \tfrac{m^2}{12}(\mathrm{Tr}\,\underset{\sim}{X})(\mathrm{Tr}\,\underset{\sim}{X}^3) + \tfrac{m^2}{32}(\mathrm{Tr}\,\underset{\sim}{X}^2)^2 \\
&+ \tfrac{m^3}{32}(\mathrm{Tr}\,\underset{\sim}{X})^2(\mathrm{Tr}\,\underset{\sim}{X}^2) + \tfrac{m^4}{384}(\mathrm{Tr}\,\underset{\sim}{X})^4\}\Delta^4 + \ldots\ldots]\; G_\rho(\theta),
\end{aligned} \qquad (14)$$

Now, using Taylor's expansion for $\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon}$, we can represent $\underset{\sim}{X}$ by

$$\underset{\sim}{X} = \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\epsilon})^{-1} - \underset{\sim}{I}$$

$$= \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \Sigma\, \epsilon_{rs}\underset{\sim}{A}^{-1}_{rs})^{-1} - \underset{\sim}{I}$$

$$= B^{-1}(I + \Sigma\, \epsilon_{rs} A\, A^{-1}_{rs})^{-1} A - I$$

$$= B^{-1}(I - \Sigma\, \epsilon_{rs} (A_{rs}\, A) + \Sigma\, \epsilon_{rs}\epsilon_{tu} (A^{-1}_{rs}\, A)(A^{-1}_{tu} A)$$

$$- \Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw} (A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A)$$

$$+ \Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\epsilon_{xy} (A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A)(A^{-1}_{xy}\, A)$$

$$- \ldots\ldots)\, A - I$$

$$= (B^{-1}A - I) - \Sigma\, \epsilon_{rs} (B^{-1}A)(A^{-1}_{rs}\, A)$$

$$+ \Sigma\, \epsilon_{rs}\epsilon_{tu} (B^{-1}A)(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)$$

$$- \Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw} (B^{-1}A)(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A)$$

$$+ \Sigma\, \epsilon_{rs}\epsilon_{tu}\epsilon_{vw}\epsilon_{xy} (B^{-1}\, A)(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A)(A^{-1}_{xy}\, A) \qquad (15)$$

where A^{-1}_{rs} is pxp matrix obtained by operating ∂_{rs} on A^{-1}; i.e., it has its (i,j)-th element as $\frac{1}{2}(\delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj})$. Now,using the notations

$$\mathrm{Tr}(A^{-1}_{rs}\, A) = (rs),$$

$$\mathrm{Tr}(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A) = (rs|tu),$$

$$\mathrm{Tr}(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A) = (rs|tu|vw),$$

$$\mathrm{Tr}(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A)(A^{-1}_{xy}\, A) = (rs|tu|vw|xy),$$

$$\mathrm{Tr}\ F\ (A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A) = (F|rs|tu),$$

$$\mathrm{Tr}(B^{-1}\, A)(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A) = (I + F|rs|tu),$$

$$\mathrm{Tr}(B^{-1}\, A)(A^{-1}_{rs}\, A)(A^{-1}_{tu}\, A)(A^{-1}_{vw}\, A) = (I + F|rs|tu|vw),$$

$$\mathrm{Tr}\ F = (F)$$

$$\mathrm{Tr}\ F^2 = (F|F)\ldots \text{ or alternatively,}$$

$$\mathrm{Tr}\ F^s = (Fs)\ldots\text{etc.}$$

and substituting Eq (15) into Eq (14), we get from term by term comparison between two expansions of J, Eqs (13) and (14) after substitution, the following:

$$
\begin{aligned}
\partial_{rs} Pr(m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta) = -\{&(\tfrac{m}{2})(\underset{\sim}{I}+\underset{\sim}{F}|rs)\Delta + [(\tfrac{m}{2})(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs) \\
&+ (\tfrac{m^2}{4})(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs)]\Delta^2 + [(\tfrac{m}{2})(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs) \\
&+ (\tfrac{m^2}{8})\ \{(\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs) + (2)(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs)\} + \tfrac{3m^3}{48}(\underset{\sim}{F})^2(\underset{\sim}{I}+\underset{\sim}{F}|rs)]\Delta^3 \\
&+ [(\tfrac{m}{2})(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs) + (\tfrac{m^2}{24})\{(6)(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs) \\
&+ (2)(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs) + (3)(\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs)\} \\
&+ (\tfrac{m^3}{16})\{(\underset{\sim}{F})^2(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs) + (\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs)\} + (\tfrac{4m^4}{384})(\underset{\sim}{F})^3(\underset{\sim}{I}+\underset{\sim}{F}|rs)]\Delta^4 \\
&+ \cdots\}\ G_\rho(\theta),
\end{aligned}
$$

Similarly expressions for $\partial_{rs}\partial_{tu}$ $Pr(m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta)$, $\partial_{rs}\partial_{tu}\partial_{vw}$ $Pr(m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta)$ **and** $\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}$ $Pr(m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta)$ **are available in** [9].

3. AN ASYMPTOTIC EXPANSION FOR PERCENTILES OF $T = m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ UP TO $O(n^{-1})$.

Now recall that $\Delta G_\rho(\theta) = -Eg_\rho(\theta)$, where $g_\rho(\theta) = [\Gamma(\rho)]^{-1}\ e^{-\theta}\ \theta^{\rho-1}$, and $\rho = mp/2$. Also we note here that

$$
E^i\ g_\rho(\theta) = \frac{\theta^i}{\rho(\rho+1)\cdots(\rho+i-1)}\ g_\rho(\theta)\,. \tag{16}
$$

Thus, it is possible to write $\partial_{st}\partial_{ur}$ $Pr[m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta]$ **in the following form:**

$$
\partial_{st}\partial_{ur}\ Pr[m\ Tr\ \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta] = -\sum_{J=1}^{4} A_J'\ E^j\ g_\rho(\theta), \tag{17}
$$

where A_j''s, $j = 1,\ldots,4$, are also available in [9]. Also, we note

$$\left.\begin{aligned}
&\sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(st|ur) = \frac{1}{2}\,p(p+1),\\
&\sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(st)(ur) = p,\quad \sum_{s,t} \sigma_{st}(st) = p,\\
&U = \sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(\underset{\sim}{F}|st)(ur) = (\underset{\sim}{F}),\\
&\sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(\underset{\sim}{F}|st|ur) = (\underset{\sim}{F})(p+1)/2,\\
&V = \sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(\underset{\sim}{F}|\underset{\sim}{F}|st|ur) = \frac{1}{2}(\underset{\sim}{F}2)(p+1),\\
&W = \sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,(\underset{\sim}{F}|rs|\underset{\sim}{F}|tu) = \frac{1}{2}((\underset{\sim}{F})^2 + (\underset{\sim}{F}2)),\ldots
\end{aligned}\right\} \tag{18}$$

,...etc.

As a check for the above relationships, let $\underset{\sim}{F}(p\times p) = \underset{\sim}{I}(p\times p)$. Thus U should be equal to p, which is the value of $\Sigma\,\sigma_{rs}\sigma_{tu}(st)(ur)$. Similarly V and W should equal $\frac{1}{2}\,p(p+1)$, which is the value of $\Sigma\,\sigma_{rs}\sigma_{tu}(st|ur)$. With the aid of these results we can evaluate A_j''s, $j = 1,\ldots,4$, after summing over all subscripts s, t, u, r.

Now by using Eqs (11), (16), (17) and (18), and putting $2\theta = y$ we get

$$h_1(\underset{\sim}{A}^{-1}) = \frac{1}{n}\left[\sum_{j=1}^{4} A_j y^j\right] g_\rho(\theta)\,[G'(\theta)]^{-1}, \tag{19}$$

where

$A_j = \sum_{r,s,t,u} \sigma_{rs}\sigma_{tu}\,A_j'/(mp)(mp+2)\ldots(mp+2j-2)$, $j = 1,\ldots,4$, and A_j coefficients have been evaluated and are available in [9].

Then we get from Eq (8) the following:

$$T = m \text{ Tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} = y + 2h_1(\underset{\sim}{A}^{-1}) + O(n^{-2})$$

$$= y + \frac{2}{n}\left[\sum_{j=1}^{4} A_j y^j\right] g_\rho(\theta)[G'(\theta)]^{-1} + O(n^{-2}). \tag{20}$$

Hence we have the following theorem:

Theorem 1 Let $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ be independently distributed $W(m,p,\underset{\sim}{B}^{-1})$, $W(n,p,\underset{\sim}{A}^{-1})$, respectively, and let $|Ch_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, where $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I}+\underset{\sim}{F}$. Then an asymptotic expansion for the percentile of $T = m \text{ Tr } \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ is given by Eq (20). The following are special cases of Eq 20.

Case 1. When terms involving $f_{ij}f_{kl}$ are negligible, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$, terms like $(\underset{\sim}{F})^2$, $(\underset{\sim}{F}2)$, $(\underset{\sim}{F})(\underset{\sim}{F}2)$,...etc. drop out. Consequently A_4 will disappear and A_1, A_2 and A_3 will be reduced to the following,

$$\left.\begin{aligned}
A_1 &= (1/4)((p+1) - m((\underset{\sim}{F})(p+1)/2 + 1) + (m^2/2)(\underset{\sim}{F})), \\
A_2 &= (1/4)((p+1) + m(1 - 2(\underset{\sim}{F})/p) - m^2(\underset{\sim}{F}))/(mp+2) \text{ and} \\
A_3 &= (1/4)((2)(\underset{\sim}{F})(p+1)/p + m((\underset{\sim}{F})(p+1)/2 + 2(\underset{\sim}{F})/p) \\
&\quad + (m^2/2)(\underset{\sim}{F}))/(mp+2)(mp+4),
\end{aligned}\right\} \tag{21}$$

and from Eq (19) we get

$$h_1(\underset{\sim}{A}^{-1}) = \frac{1}{n}\left[\sum_{j=1}^{3} A_j y^j\right] g_\rho(\theta)[G'(\theta)]^{-1} \tag{22}$$

which agrees with C-P [1] to the indicated order after simplification.

Case 2. As defined earlier,

$$y = \sum_{j=1}^{p} \lambda_j X_j^2(m)$$

where $X_j^2(m)$'s are independent central chi-square variables with m d.f. and λ_j's are ch. roots of $\underset{\sim}{U} = \underset{\sim}{A}\,\underset{\sim}{B}^{-1}$. Another check can be made by putting $\underset{\sim}{F}(pxp) = \underset{\sim}{0}(pxp)$. Then

$$y = X^2(mp)$$

is a central chi-square variable with mp d.f. and $G(\theta) = G_\rho(\theta)$. Hence we get

$$T = X^2 + \frac{1}{2n}\left[((p+m+1)/(mp+2))X^4 + (p-m+1)X^2\right] + 0(n^{-2})$$

where $X^2 = X^2(mp)$. This agrees with Ito's result [5] to the indicated order.

4. AN ASYMPTOTIC EXPANSION FOR THE C.D.F. OF $T = m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ UP TO $0(n^{-1})$

In this section we will derive an asymptotic expansion for the c.d.f. of T to $0(n^{-1})$, following the method described earlier. Again we write

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\} = \int_R \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta | \underset{\sim}{S}_2\}\Pr\{d\underset{\sim}{S}_2\}$$

$$= \Theta \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\}.$$

From Eq (7) we get

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\} = G(\theta) + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}G(\theta) + 0(n^{-2})$$

$$= G(\theta) - \frac{1}{n}[h_1(\underset{\sim}{A}^{-1})]G'(\theta) + 0(n^{-2}). \qquad (23)$$

Let $F(2\theta)$ be the c.d.f. of T; i.e.,

$$F(2\theta) = \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\}.$$

Upon substituting Eq (19) in Eq (23) and replacing y by 2θ we get

$$F(2\theta) = G(\theta) - \frac{1}{n}\left[\sum_{j=1}^{4} A_j(2\theta)^j\right] g_\rho(\theta) + 0(n^{-2}). \qquad (24)$$

Hence we have the following theorem:

Theorem 2. **Under the assumption (i) of Theorem 1, the asymptotic expansion for c.d.f. of T is given by** Eq (24).

Special cases:

1. **Upon neglecting all terms involving $f_{ij}f_{kl}$, where f_{ij} is the (ij) element of $\underset{\sim}{F}$, we get from** Eq (24)

$$F(2\theta) = G(\theta) - \frac{1}{n}\left[\sum_{j=1}^{3} A_j(2\theta)^j\right] g_\rho(\theta) + O(n^{-2}),$$

where A_j's, $j = 1,2,3$ are the same as in Eq (21), **and this agrees with C-P [1] to the indicated order after some simplifications.**

2. **Again we put $\underset{\sim}{F}(pxp) = \underset{\sim}{0}(pxp)$, and we get**

$$F(2\theta) = G_\rho(\theta) - \frac{1}{2n}[(p-m+1)\theta + 2((p+m+1)/(mp+2))\theta^2]\, g_\rho(\theta) + O(n^{-2}),$$

and this agrees with Ito's result [5] to the indicated order.

5. AN ASYMPTOTIC EXPANSION FOR PERCENTILES OF $T = m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ TO $O(n^{-2})$

Here, using the technique stated earlier, we obtain the terms of order n^{-2}. The results of the third and fourth derivatives of $G(\theta)$ as it stands are not convenient for practical use. In order to make some simplifications we assume that terms involving $f_{ij}f_{kl}$ are negligable, where f_{ij} is the (i,j) element of the deviation matrix $\underset{\sim}{F}$. From the third and fourth derivatives given in [9] and from Eq (12), using a technique similar to that of [5], we get after tedious simplifications

$$h_2(A^{-1}) = \frac{1}{48n^2}\sum_{j=1}^{4} B_j' y^j \; g_\rho(\theta)[G'(\theta)]^{-1} - \frac{1}{8n^2}\sum_{j=1}^{5} C_j y^j g_\rho(\theta)[G'(\theta)]^{-1}, \quad (25)$$

where

$$B_j' = b_j' + 24(C_j^{(2)}/2^j) - 64(C_j^{(1)}/2^j), \; j = 1,\ldots,4. \quad (26)$$

The coefficients $C_j^{(1)}$ and $C_j^{(2)}$ are given in Eq (32), but the b_j''s and C_j's are listed below.

$$
\left.
\begin{aligned}
b_1' &= 7p^2 + (-12m+12)p + (7m^2-12m+1), \\
b_2' &= (13p^2+24p-11m^2+7)/(mp+2), \\
b_3' &= (4mp^3+2(3m^2+3m+10)p^2+2(2m^3+3m^2+17m \\
&\quad + 18)p + 4\ (5m^2+9m+2))/(mp+2)^2(mp+4), \\
b_4' &= 6(p-1)(p+2)(m-1)(m+2)/(mp+2)^2(mp+4)(mp+6), \\
C_1 &= (\underset{\sim}{F})((p-m+1)((m^2/8)p(p+1) - (m^3/8)p) + (-(m/4)p. \\
&\qquad (p+1) + (m^2/4)p)(-(m/2)(p+1) + (m^2/2))), \\
C_2 &= (\underset{\sim}{F})((p-m+1)(-(m/8)(p+1) + (m/2)/p + (3m^2/8)) \\
&\quad + (p+m+1)((m^2/8)p(p+1) - (m^3/8)p) + (-(m/4)p. \\
&\qquad (p+1) + (m^2/4)p)(-(2m)/p-m^2)/(mp+2) - (m/2)(-(m/2) \\
&\qquad (p+1) + (m^2/2))), \\
C_3 &= (\underset{\sim}{F})((p-m+1)(-(p+1)/2p - (m/8)(p+1)-m/p-(3m^2/8))/ \\
&\qquad (mp+2) + (p+m+1)(-(m/8)(p+1) + (m/2)/p + (3m^2/8))/ \\
&\qquad (mp+2) + (p-m+1)(-(m/8)(p+1) + (m/2)/p + (3m^2/8)) \\
&\quad + (-m/2)(-(2m)/p - (m^2))/(mp+2) + (-(m/2)(p+1) \\
&\quad + (m^2/2))((p+1)/4 + (m/4)p)/(mp+2)), \\
C_4 &= (\underset{\sim}{F})((p-m+1)((p+1)/2p + (m/8)(p+1) + (m/2)/p \\
&\quad + (m^3/8))/(mp+2)(mp+4) + (p+m+1)(-(p+1)/2p \\
&\quad - (m/8)(p+1) - (m/p) - (3m^3/8))/(mp+2)^2 + (-m/2) \\
&\qquad (2(p+1)/p + m((p+1)/2 + (2/p))+m^2)/(mp+2)(mp+4) \\
&\quad + ((p+1)/4 + (m/4)p)(-(2m)/p - m^2))/(mp+2)^2)
\end{aligned}
\right\} \quad (27)
$$

and

$$C_5 = (\underset{\sim}{F})((p+m+1)((p+1)/2p + (m/8)(p+1) + (m/2)/p + (m^3/8))/(mp+2)^2(mp+4) + ((p+1)/4 + (m/4)p)(2(p+1)/p + m((p+1)/2 + 2/(p) + (m^2/2))/(mp+2)^2(mp+4).$$

Now we substitute Eqs (21) and (25) into Eq (8) to get

$$T = m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} = y + 2h_1(\underset{\sim}{A}^{-1}) + 2h_2(\underset{\sim}{A}^{-1}) + 0(n^{-3})$$

$$= y + \frac{2}{n} \sum_{j=1}^{3} A_j y^j g_p(\theta)[G'(\theta)]^{-1} + \frac{1}{24n^2} \sum_{j=1}^{4} B_j' y^j .$$

$$g_p(\theta)[G'(\theta)]^{-1} - \frac{1}{4n^2} \sum_{j=1}^{5} C_j\, y^j\, g_p(\theta)[G'(\theta)]^{-1} + 0(n^{-3}) \qquad (28)$$

where A_j's are given by Eq (20). The B_j''s and C_j's are given above, and we get as a final result the following:

<u>Theorem 3</u>. Let $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ be independently distributed $W(m,p,\underset{\sim}{B}^{-1})$ and $W(n,p,\underset{\sim}{A}^{-1})$, respectively, and let

(i) $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I}+\underset{\sim}{F}$ and $|\mathrm{Ch}_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, and

(ii) terms involving $f_{ij}f_{kl}$ be negligable, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$. Then the asymptotic expansion for the percentile of $T = m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ is given by Eq (28).

6. AN ASYMPTOTIC EXPANSION FOR THE C.D.F. OF $T = m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ TO ORDER n^{-2}

Here we will derive an asymptotic expansion for the c.d.f. of T following the methods described in the previous pages. Again we write

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\} = \int_R \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} \le 2\theta\} \Pr\{d\underset{\sim}{S}_2\}$$

$$= \theta \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1\underset{\sim}{A} \le 2\theta\}.$$

From Eq (7) we have

$$\Pr\{m \operatorname{Tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \le 2\theta\} = G(\theta) + \frac{1}{n} \Sigma\, \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} G(\theta) + \frac{4}{3n^2} \Sigma\, \sigma_{rs}\sigma_{tu}\sigma_{vw}\partial_{st}\partial_{uv}\partial_{wr} G(\theta) + \frac{1}{2n^2} \Sigma\, \sigma_{rs}\sigma_{tu}\sigma_{vw}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{yv} G(\theta) + O(n^{-3}). \tag{29}$$

Let $F(2\theta)$ be the c.d.f. of T, i.e.,

$$F(2\theta) = \Pr\{m \operatorname{Tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \le 2\theta\}.$$

Upon substituting Eq (19) and results from [9] we get after tedious simplifications the following:

$$F(2\theta) = G(\theta) - \frac{1}{n}\left[\sum_{j=1}^{4} A_j\,(2\theta)^j\right] g_\rho(\theta) - \frac{1}{48n^2}\left[\sum_{j=1}^{4} B_j\,(\theta)^j\right] g_\rho(\theta) + O(n^{-3}) \tag{30}$$

and

$$B_j = b_j - 64C_j^{(1)} + 24C_j^{(2)}, \tag{31}$$

where the coefficients A_j's, are available in [9] and the rest of the coefficients are listed below:

$$
\left.
\begin{aligned}
b_1 &= -(3mp^3 - 2(3m^2-3m+4)p^2 + 3(m^3-2m^2+5m-4)p - 8m^2+12m+4) \;, \\
b_2 &= -2(3mp^3+2(3m^2+3m-4)p^2 -3(3m^3-2m^2-5m+4)p - 8m^2+12m+4)/(mp+2) \;, \\
b_3 &= 4(3mp^3 - 2(3m^2-3m-4)p^2 - 3(3m^3+2m^2+11m-4)p - 40m^2 - 36m - 4)/ \\
&\quad (mp+2)(mp+4) \;, \\
b_4 &= 24(mp^3+2(m^2+m+4)p^2 + (m^3+2m^2+21m+20)p + (8m^2+20m+20))/ \\
&\quad (mp+2)(mp+4)(mp+6) \;, \\
c_1^{(1)} &= (\underset{\sim}{F})(-(m/16)(p^2+3p+4) + (3m^2/8)(p+1) - m^3/8) \;, \\
c_2^{(1)} &= (\underset{\sim}{F})((-3/2)(p^2+3p+4)/p - (3m/2)(p+1)/p \\
&\quad - (3m^2/4)(p+1) + (3m^2/2)/p + (3m^3/4))/(mp+2) \;, \\
c_3^{(1)} &= -(\underset{\sim}{F})((6m)(p+1)/p + (3m^2/2)(p+1) + (6m^2)/p \\
&\quad + (3m^3/2))/(mp+2)(mp+4) \;, \\
c_4^{(1)} &= (\underset{\sim}{F})((6)(p^2+3p+4)/p + (m/2)(p^2+3p+4) \\
&\quad + (18m)(p+1)/p + (3m^2)(p+1) + (6m^2)/p \\
&\quad + (m^3))/(mp+2)(mp+4)(mp+6) \;, \\
c_1^{(2)} &= -(\underset{\sim}{F})(2(2p^2+5p+5)/p + (m/2)(2p^2+5p+5) \\
&\quad + (8m)(p+1)/p + (m)(p^3+2p^2+3p+2)/p + (m^2) \\
&\quad (p^2+p+4)/p + (m^3/4)(p^2+p+4) + (m^3/2)) \;, \\
c_2^{(2)} &= (\underset{\sim}{F})((16)(2p^2+5p+4)/p + (2m)(2p^2+5p+5) \\
&\quad + (60m)(p+1)/p + (6m)(p^3+2p^2+3p+2)/p \\
&\quad + (6m^2)(p+1) + (m^2)(p^3+2p^2+3p+2) + (9m^2) \\
&\quad (p^2+p+4)/p + (3m^3/2)(p^2+p+4) + (3m^3))/(mp+2) \;, \\
c_3^{(2)} &= -(\underset{\sim}{F})((48)(2p^2+5p+5)/p + (8m)(2p^2+5p+5) \\
&\quad + (192m)(p+1)/p + (28m)(p^3+2p^2+3p+2)/p \\
&\quad + (24m^2)(p+1) + (4m^2)(p^3+2p^2+3p+2) + (24m^2) \\
&\quad (p^2+p+4)/p + (3m^3)(p^2+p+4) + (6m^3))/(mp+2)\;(mp+4)
\end{aligned}
\right\} \quad (32)
$$

and

$$C_4^{(2)} = (\underset{\sim}{F})((64)(2p^2+5p+5)/p + (8m)(2p^2+5p+5)$$
$$+ (208m)(p+1)/p + (32m)(p^3+2p^2+3p+2)/p$$
$$+ (24m^2)(p+1) + (4m^2)(p^3+2p^2+3p+2) + (20m^2)$$
$$(p^2+p+4)/p + (2m^3)(p^2+p+4) + (4m^3))/(mp+2)$$
$$(mp+4)(mp+6).$$

Then, we have the following final form:

<u>Theorem 4</u>. Under the same assumptions of Theorem 1, the asymptotic expansion for the c.d.f. of T to $O(n^{-2})$ is given by Eqs (30), (31) and (32).

7. NUMERICAL STUDY OF POWERS AND ACCURACY COMPARISONS

To show how the accuracy has improved by introducing the terms involving $f_{ij}f_{kl}$, where f_{ij} is the (i,j) element of the deviation matrix F, as well as terms of order n^{-2}, some numerical results are presented in this section. Some comparisons may be made from Table I between the exact and approximate powers of T when p=3 and m=4 for n=34 and n=84. Values of the exact powers in the table are taken from Pillai and Sudjana [4]. Our expansion, which is given above by Eqs (30), (31) and (32), is used for the computation of this table up to $O(n^{-1})$ and $O(n^{-2})$. To illustrate the usefulness of the neglected terms in the C-P approximation [1] values using their approximation are given in ( ) in the table. It may be observed that their values differ considerably from the exact while our improvement leaves practically very little error. One can see from Table I that the approximation given by Chattopadhyay-Pillai to order 1/n is not very good even after adding terms of order $1/n^2$. Again from Table I it is obvious that the accuracy given by terms of order n^{-1} is not enough even after including those $f_{ij}f_{kl}$ terms, and the usefulness of terms of order n^{-2} is also considerable. Further power computations have been carried out for p=3 and p=4 and presented in [9]. For tabulations of powers, the upper five percent points were taken from Davis [10].

TABLE I

THE COMPARISON BETWEEN THE EXACT AND APPROXIMATE POWERS OT T-TEST FOR $p=3$, $m=4$, $\alpha=0.05$ AND FOR EQUAL DEVIATION PARAMETERS.

($\underset{\sim}{\Gamma}$)	Up to the order	$n=34$	$n=84$
0.001	$0(1)$	0.013	0.031
	$0(n^{-1})$	0.036	0.048
	$0(n^{-2})$	0.049	0.0501
	Exact	0.050	0.0501
0.150	$0(1)$	0.019	0.043
	$0(n^{-1})$	0.049 (0.052)	0.063 (0.065)
	$0(n^{-2})$	0.064	0.065
	Exact	0.064	0.066
0.500	$0(1)$	0.040	0.079
	$0(n^{-1})$	0.091 (0.097)	0.108 (0.115)
	$0(n^{-2})$	0.107	0.109
	Exact	0.102	0.109
1.000	$0(1)$	0.087	0.150
	$0(n^{-1})$	0.179 (0.178)	0.188 (0.206)
	$0(n^{-2})$	0.202	0.189
	Exact	0.173	0.189

The figures in ( ) are computed using Chattopadhyay-Pillai expansion.

SECTION II

ASYMPTOTIC FORMULAE FOR THE PERCENTILE AND C.D.F. OF HOTELLING'S TRACE UNDER VIOLATIONS

1. INTRODUCTION

In the previous section, asymptotic expansions for the distribution and percentile of the statistic $T = m \operatorname{Tr} S_1 S_2^{-1}$ have been obtained up to terms of the order $1/n^2$, where mS_1 and nS_2 are independently distributed central Wishart with m degrees of freedom and covariance matrix $\Sigma_1, W(m,p,\Sigma_1)$, and with n degrees of freedom and covariance matrix Σ_2, $W(n,p,\Sigma_2)$, respectively. Further, denoting the non-centrality by $(F) = \operatorname{Tr}F = \operatorname{Tr}(B^{-1}A-I)$, where $\Sigma_1^{-1} = B$ and $\Sigma_2^{-1} = A$, we also included terms involving $f_{ij}f_{kl}$, where f_{ij} is the (i,j)-th element of F, which were previously neglected by Chattopadhyay and Pillai [1]. In this section again we extend the work of Chattopadhyay [11], who derived an asymptotic expansion up to terms of order 1/n, neglecting $f_{ij}f_{kl}$ terms for c.d.f. and percentile of the trace statistic when mS_1 has non-central Wishart distribution with m degrees of freedom, covariance matrix Σ_1 and non-centrality parameter Ω, $W(m,p,\Sigma_1,\Omega)$ and nS_2 distributed central Wishart $W(n,p,\Sigma_2)$. The extension in this case is to include the $f_{ij}f_{kl}$ terms neglected by him. It may be noted that these terms were found to improve the expansion in the previous section. The results are helpful for the study

of the violation of a) the assumption of a common covariance matrix in the MANOVA test based on the trace statistic and b) the normality assumption in testing $\Sigma_1 = \Sigma_2$. For $\Sigma_1 = \Sigma_2$, asymptotic expansions of the non-central c.d.f. have been studied by several authors [12] and [13].

2. THE METHOD OF ASYMPTOTIC EXPANSION

The notations in this section generally follow those of the previous section and other papers referred to earlier [1], [5], but additional notations will be introduced here. The method herein is also to obtain an asymptotic expansion for the percentile of T first, and use it to derive an expansion for the c.d.f. of T, where T may be defined as follows:

Let $Z = (z_1,\dots,z_m)$ be a pxm matrix of independently distributed columns vectors,where z_i has the density $N(\mu_i,\Sigma_i)$, $i = 1,\dots,m$. Then we may define $T = \mathrm{Tr}\, S_2^{-1}ZZ' = \sum_{i=1}^{m} z_i'S_2^{-1}z_i$ where nS_2 is distributed $W(n,p,\Sigma_2)$ independently of Z.

Now,if S_2^{-1} is replaced by B in T, then Tr BZZ' is distributed as a non-central chi-square with mp degrees of freedom and non-centrality parameter ω^2, where

$$\omega^2 = \mathrm{Tr}\, BMM' = \mathrm{Tr}\, \Omega$$

$$M = \{\mu_1,\dots,\mu_m\} \neq 0, \ \rho = mp/2$$

we may note that

$$\Pr\{\mathrm{Tr}\, BZZ' \leq \theta\}$$

$$= e^{-\omega^2/2} \sum_{J=0}^{\infty} \frac{(\omega^2/2)^J}{J!\,2^{\rho+J}\,\Gamma(\rho+J)} \int_0^{\theta} x^{\rho+J-1} e^{-x/2} dx$$

$$= G_{mp}(\theta,\omega^2),$$

where $G_{mp}(\theta,\omega^2)$ is the c.d.f. of non-central chi-square with mp degrees of freedom and the non-centrality parameter ω^2.

Let

$$G(\theta) = \Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}.$$

As a first approximation, for large n we may replace $\underset{\sim}{A}^{-1}$ by $\underset{\sim}{S}_2$ in $G(\theta)$, and consider

$$G(\theta) = \Pr\{\mathrm{Tr}\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}.$$

Furthermore, as suggested by Ito [5], obtain a function $h(\underset{\sim}{S}_2)$ of the elements of $\underset{\sim}{S}_2$ and n large enough such that

$$G(\theta) = \Pr\{\mathrm{Tr}\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)\} \tag{33}$$

and then write $h(\underset{\sim}{S}_2)$ as a series with the first term being a linear function of non-central chi-square variables and terms of decreasing order of magnitude,

Now Eq (33) can be written such that

$$G(\theta) = E_{\underset{\sim}{S}_2}\{\Pr[\mathrm{Tr}\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)/\underset{\sim}{S}_2]\}. \tag{34}$$

By using Taylor's expansion it is possible to expand $\Pr[\mathrm{Tr}\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2]$ about an origin $(\sigma_{11},\ldots,\sigma_{pp},\sigma_{12},\ldots,\sigma_{p-1,p})$, where

$$\underset{\sim}{A}^{-1} = (\sigma_{ij}) \ , \quad (i,j) = (1,\ldots,p). \tag{35}$$

Thus,

$$\Pr\{\mathrm{Tr}\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \le h(\underset{\sim}{S}_2)\,|\,\underset{\sim}{S}_2\}$$

$$= \{\exp[\mathrm{Tr}(\underset{\sim}{S}_2-\underset{\sim}{A}^{-1})\underset{\sim}{\partial}]\Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \le h(\underset{\sim}{A}^{-1})\}, \tag{36}$$

where

$$\underset{\sim}{\partial}(p\mathrm{x}p) = (\tfrac{1}{2}(1+\delta_{ij})\ \frac{\partial}{\partial\sigma_{ij}}) \tag{37}$$

and δ_{ij} is the Kronecker delta. Hence

$$G(\theta) = \Theta \cdot \Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \le h(\underset{\sim}{A}^{-1})\} \tag{38}$$

where

$$\Theta = \exp[-\mathrm{Tr}\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial}]\,|\underset{\sim}{I} - \tfrac{2}{n}\underset{\sim}{A}^{-1}\underset{\sim}{\partial}|^{-(n/2)}$$

$$= 1 + \frac{1}{n}\sum\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + O(n^{-2}) \tag{39}$$

where Σ denotes the summation over all suffixes r,s,..., each of which range from 1 to p. Expanding $h(\underset{\sim}{S}_2)$ around θ gives

$$h(\underset{\sim}{S}_2) = \theta + h_1(\underset{\sim}{S}_2) + h_2(\underset{\sim}{S}_2)+\ldots \tag{40}$$

where $h_s(\underset{\sim}{S}_2)$ is $O(n^{-s})$. From Eqs (38) and (39) and expanding $h(\underset{\sim}{S}_2)$ around $h(\underset{\sim}{A}^{-1})$ we can get

$$G(\theta) = [1 + \frac{1}{n}\sum\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + O(n^{-2})][\ 1 + h_1(\underset{\sim}{A}^{-1})D + O(n^{-2})]$$

$$\Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \le \theta\},$$

where $D = \frac{\partial}{\partial\theta}$, and by equating terms of successive order we get

$$[h_1(\underset{\sim}{A}^{-1})D + \frac{1}{n}\sum\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}]\Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \le \theta\} = 0. \tag{41}$$

For the purpose of evaluating $\partial_{st}\partial_{ur}\ \Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \le \theta\}$ we will use the perturbation technique [8].

Let

$$J = \Pr\{\mathrm{Tr}(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} \underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} \tag{42}$$

where $\underset{\sim}{\varepsilon}$(pxp) is a symmetric matrix sufficiently close to $\underset{\sim}{0}$(pxp).
Following [11], [5] and [13], we get

$$J = |\underset{\sim}{I} - \underset{\sim}{X}\Delta|^{-(m/2)} \mathrm{Exp}[-\omega^2/2]$$

$$\mathrm{Exp}\{(1/2)E\ \mathrm{Tr}(\underset{\sim}{I} - \underset{\sim}{X}\Delta)^{-1}\underset{\sim}{\Omega}\}G_{mp}(\theta,0) \tag{43}$$

where $\Delta = E-1$, $E^r G_{mp}(\theta,\omega^2) = G_{mp+2r}(\theta,\omega^2)$ and

$$\begin{aligned} \underset{\sim}{X} &= \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I} \\ &= (\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I}) - \sum \varepsilon_{rs}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A}) \\ &+ \sum \varepsilon_{rs}\varepsilon_{tu}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) - \ldots \end{aligned}$$

where $\underset{\sim}{A}_{rs}^{-1}$ is the pxp matrix with (i,j)-th element $(1/2)(\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})$.
Also by Taylor's theorem J can be expressed in the following form

$$J = \{1 + \sum \varepsilon_{rs}\partial_{rs} + \frac{1}{2!}\sum \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} + \ldots\}$$
$$\Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} \tag{44}$$

Now, if $\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I} = \underset{\sim}{F}$ such that $|ch_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, upon using the notations

$$\begin{aligned} &\mathrm{Tr}(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A}) = (rs) \\ &\mathrm{Tr}(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) = (rs|tu) \\ &\mathrm{Tr}(\underset{\sim}{F})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) = (\underset{\sim}{F}|rs|tu) \\ &\mathrm{Tr}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) = (\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ &\mathrm{Tr}(\underset{\sim}{F}^2) = (\underset{\sim}{F}^2),\ \mathrm{Tr}(\underset{\sim}{F}^s) = (\underset{\sim}{F}^2),\ldots \text{ etc.} \end{aligned}$$

and substituting $\underset{\sim}{X}$ in Eq (43) term by term comparison between the two expansions of J, Eqs (44) and (43), after substituting $\underset{\sim}{X}$ will give the second derivative $\partial_{st}\partial_{ur}\Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$, which can be written in the following form:

$$\partial_{st}\partial_{ur} \Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = 2\cdot \sum_{J=0}^{6} A_J' E^J G_{mp}(\theta,\omega^2) \tag{45}$$

where

$$\begin{aligned} A_0' = {} & (\tfrac{m}{4})\{-2(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) + (\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & + 2(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) - 2(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & - 2(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu)\} + (\tfrac{m^2}{8})\{(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & + 2(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) - (\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) - 2(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) \\ & - 2(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|tu) - (\underset{\sim}{F}^2)(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu)\} \\ & - (\tfrac{m^3}{16})\{(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{I}+\underset{\sim}{F}|tu) + (\underset{\sim}{F})^2(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu)\} . \end{aligned}$$

Other A_j''s coefficients are available in [14], [15].

3. AN ASYMPTOTIC EXPANSION FOR THE PERCENTILE OF $T = \mathrm{Tr}\, \underset{\sim}{S}_2^{-1} \underset{\sim}{Z}\underset{\sim}{Z}'$

Recalling that $G_{mp}(\theta,\omega^2)$ is the c.d.f. of the non-central chi-square distribution with mp degrees of freedom and non-centrality parameter ω^2, we may note that

$$E^r G_{mp}(\theta,\omega^2) = G_{mp+2r}(\theta,\omega^2).$$

Hence, it is possible to rewrite Eq (45) in the following form,

$$\partial_{st}\partial_{ur}\Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = 2 \sum_{J=0}^{6} A_J' G_{mp+2J}(\theta,\omega^2) . \tag{46}$$

Again, we note

$$\sum_{r,s,t,u} \sigma_{st}\sigma_{ur}(rs|tu) = \frac{1}{2}\,p(p+1),$$

$$\sum\sigma_{rs}(rs) = p,\ \sum\sigma_{st}\sigma_{ur}(rs)(tu) = p,$$

$$u = \sum\sigma_{st}\sigma_{ur}(\underset{\sim}{F}|rs)(tu) = (\underset{\sim}{F}),$$

$$v = \sum\sigma_{st}\sigma_{ur}(\underset{\sim}{F}|\underset{\sim}{F}|rs|tu) = \frac{1}{2}(\underset{\sim}{F}^2)(p+1),$$

$$w = \sum\sigma_{st}\sigma_{ur}(\underset{\sim}{\Omega}|rs|\underset{\sim}{F}|tu) = \frac{1}{2}[(\underset{\sim}{\Omega})(\underset{\sim}{F}) + (\underset{\sim}{\Omega}\underset{\sim}{F})],\ldots\text{etc.}$$

As a check for the above relationships, let $\underset{\sim}{F}(pxp) = \underset{\sim}{I}(pxp)$ and $\underset{\sim}{\Omega}(pxp) = \underset{\sim}{I}(pxp)$; thus $\underset{\sim}{u}$ should equal to p,which is the value of $\sum\sigma_{st}\sigma_{ur}(rs)(tu)$. Similarly v and w will be reduced to $\sum\sigma_{st}\sigma_{ur}(rs|tu)$ equal to 1/2p(p+1): With the help of the above relationships, it is possible to evaluate the A_J''s, J = 0,...,6, after summing over all subscripts, r, s, t, u.

Now by using Eq (41) and the above coefficients we get

$$\begin{aligned}
&-h_1(\underset{\sim}{A}^{-1})D\ \Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} \\
&\quad = \frac{1}{4n}\sum_{J=0}^{4} a_J(m,p)G_{mp+2J}(\theta,\omega^2) \\
&\quad + \frac{1}{n}\sum_{J=0}^{6} A_J(m,p)G_{mp+2J}(\theta,\omega^2),
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= mp(m-p-1) \\
a_1 &= -2m(mp-\omega^2), \\
a_2 &= mp(m+p+1)-2(2m+p+1)\omega^2 + \mathrm{tr}\ \underset{\sim}{\Omega}^2, \\
a_3 &= 2\{(m+p+1)\omega^2 - \mathrm{tr}\ \underset{\sim}{\Omega}^2\}, \\
a_4 &= \mathrm{tr}\ \underset{\sim}{\Omega}^2,
\end{aligned}$$

and $A_J = \sum\limits_{r,s,t,u} \sigma_{rs}\sigma_{tu}A'_J \qquad J = 0,\ldots,6.$

The above can be simplified after tedious algebra, and it can be written in the following simple expression,

$$A_J = \sum_{k=0}^{3} m^k A_{Jk} \qquad J = 0,1,\ldots,6. \tag{47}$$

Some of the A_{Jk}'s are listed below and the rest are listed in [14]:

$$A_{00} = 0,$$

$$A_{01} = -(1/4)[2[(\underset{\sim}{F})^2+(\underset{\sim}{F}^3)](p+1)+[(\underset{\sim}{F})^2+(\underset{\sim}{F}^2)+2(\underset{\sim}{F}^2)(\underset{\sim}{F})+2(\underset{\sim}{F}^3)]],$$

$$\begin{aligned} A_{02} = {} & (1/8)[[(\underset{\sim}{F})p-2(\underset{\sim}{F})^2-3(\underset{\sim}{F})(\underset{\sim}{F}^2)-(\underset{\sim}{F}^2)^2](p+1) \\ & - [(\underset{\sim}{F})^3+(\underset{\sim}{F})(\underset{\sim}{F}^2)+6(\underset{\sim}{F}^2)+4(\underset{\sim}{F}^3)]], \end{aligned}$$

$$A_{03} = -(1/8)[(1/2)[(\underset{\sim}{F})^2p+(\underset{\sim}{F})^3](p+1)+(\underset{\sim}{F})p+2(\underset{\sim}{F})^2+(\underset{\sim}{F})(\underset{\sim}{F}^2)],$$

$$A_{10} = \{-3(\underset{\sim}{\Omega})[(\underset{\sim}{F}^2)+(\underset{\sim}{F}^3)] - 3[(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + (\underset{\sim}{\Omega}\underset{\sim}{F}^3)]\},$$

$$\begin{aligned} A_{11} = {} & \{[24(\underset{\sim}{F}^2)+18(\underset{\sim}{F}^3)+(\underset{\sim}{\Omega})(\underset{\sim}{F}^3) + [(3/2)p+3(\underset{\sim}{F}^2)+3(\underset{\sim}{F}^3)].\ (\underset{\sim}{\Omega}\underset{\sim}{F}) \\ & + [-(3/2)p + 3(\underset{\sim}{F})+3(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F}^2) + [3p+3(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}^3)](p+1) \\ & + [(9/2)(\underset{\sim}{F}^2)^2+(45/2)(\underset{\sim}{F})(\underset{\sim}{F}^2)+19(\underset{\sim}{F})(\underset{\sim}{F}^3)](\underset{\sim}{\Omega}) + [(3/2)(\underset{\sim}{F})^2 \\ & + (9/2)(\underset{\sim}{F}^2)+3(\underset{\sim}{F})(\underset{\sim}{F}^2)+4(\underset{\sim}{F}^3)](\underset{\sim}{\Omega}\underset{\sim}{F})+[6+21(\underset{\sim}{F})+(3/2)(\underset{\sim}{F})^2 \\ & + (9/2)(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F}^2)+[18(\underset{\sim}{F})+54](\underset{\sim}{\Omega}\underset{\sim}{F}^3)+36(\underset{\sim}{\Omega}\underset{\sim}{F}^4)+12(\underset{\sim}{F})^2 \\ & + 12(\underset{\sim}{F}^2)+18(\underset{\sim}{F})(\underset{\sim}{F}^2)+18(\underset{\sim}{F}^3)\}/(12), \end{aligned}$$

$$A_{12} = \{[-(\underset{\sim}{F})p+3(\underset{\sim}{F}^2)p+8(\underset{\sim}{F})^2+9(\underset{\sim}{F})(\underset{\sim}{F}^2)+[(\underset{\sim}{F})(\underset{\sim}{F}^2)-(\underset{\sim}{F}^2)$$

$$+ (F)^2](\Omega)/2+[-(F)p+2(F)^2+3(F)(F^2)+(F^2)](\Omega F)/2$$

$$+[(F)p+(F)^2](\Omega F^2)].\ (p+1)+[-2(F)+(F^2)-(1/2).$$

$$(F)(F^2)+(F^3)+2(F^2)(F)^2+(F)^3](\Omega)+[-p+2(F)$$

$$+ (9/2)(F^2)+2(F)^2+(F)(F^2)+2(F^3)+(F)^3/2]\ (\Omega F)$$

$$+ [p+12(F)+(3/2)(F)^2+2(F^2)](\Omega F^2)+6(\Omega F^3)(F)$$

$$+ 4(F)+20(F^2)+12(F^3)\}/(8),$$

$$A_{13} = \{[9(F)^2p+(F^2)(F)^2+(1/2)(\Omega)(F)^3+[(F)^2+(F)^3].\ (3(\Omega F)/2)].(p+1)$$

$$+ [3(F)^2+(1/2)(F)^4](\Omega)+[3(F)p+12(F)^2+3(F)(F^2)$$

$$+ (1/2)(F)^3](\Omega F)+3(\Omega F^2)(F)^2+18(F)p+36(F)^2$$

$$+ 18(F^2)(F)\}/(48),$$

$$A_{20} = \{-(\Omega F^2)(p+1)+[4(F)+42(F^2)+36(F^3)](\Omega)$$

$$+ [4-3(\Omega)(F)-3(\Omega)(F^2)](\Omega F)+[38-(\Omega)(F)](\Omega F^2)$$

$$- 3(\Omega F)^2+28(\Omega F^3)-4(\Omega F^4)+(\Omega F\Omega')+(\Omega' F\Omega)$$

$$+ (\Omega' F^2\Omega)-4(\Omega F)(\Omega F^2)\}/(4),$$

$$A_{21} = \{[-8(F)-40(F^2)-24(F^3)+(\Omega)[4(F)-2(F^2)-(16/3)(F^3)]$$

$$+ [-2p-20(F^2)-16(F^3)](\Omega F)+[4p-20(F)-16(F^2)](\Omega F^2)-(16)$$

$$[p+(F)](\Omega F^3)+[-p/2+(F)+(F^2)](\Omega F)^2+[2p+2(F)](\Omega F)(\Omega F^2)$$

$$+\ (\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}^2)](p+1)-[8(\underset{\sim}{F})^2+134(\underset{\sim}{F})(\underset{\sim}{F}^2)+\frac{160}{3}(\underset{\sim}{F})(\underset{\sim}{F}^3)+24(\underset{\sim}{F}^2)^2](\underset{\sim}{\Omega})$$

$$-\ [8(\underset{\sim}{F})-16+10(\underset{\sim}{F})^2+36(\underset{\sim}{F}^2)+\frac{64}{3}(\underset{\sim}{F}^3)+16(\underset{\sim}{F})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F})$$

$$-\ [64+98(\underset{\sim}{F})+32(\underset{\sim}{F}^2)+8(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}^2)-[312+96(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}^3)$$

$$-\ 192(\underset{\sim}{\Omega}\underset{\sim}{F}^4)+[4+6(\underset{\sim}{F})+(3/2)(\underset{\sim}{F}^2)+(1/2)(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F})^2$$

$$+\ [-4+6(\underset{\sim}{F})^2+7(\underset{\sim}{F})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})-20(\underset{\sim}{F})^2-20(\underset{\sim}{F}^2)-24(\underset{\sim}{F})(\underset{\sim}{F}^2)$$

$$+\ 24(\underset{\sim}{\Omega}\underset{\sim}{F}')-24(\underset{\sim}{F}^3)+[-2(\underset{\sim}{F})+(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}\underset{\sim}{\Omega}')+[6(\underset{\sim}{F})$$

$$+\ (\underset{\sim}{F}^2)](\underset{\sim}{\Omega}'\underset{\sim}{F}\underset{\sim}{\Omega})+[14(\underset{\sim}{F})+(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}'\underset{\sim}{F}^2\underset{\sim}{\Omega})+[28+6(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}).(\underset{\sim}{\Omega}\underset{\sim}{F}^2)$$

$$+\ 12(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}^3)+[-2(\underset{\sim}{F})+(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}'\underset{\sim}{\Omega})+8(\underset{\sim}{F})(\underset{\sim}{\Omega}'F^3\underset{\sim}{\Omega})$$

$$+\ 4(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F}^2)+4(\underset{\sim}{\Omega}\underset{\sim}{F}^2)^2\}/(16),$$

$$A_{22}=\{[-4(\underset{\sim}{F})p-40(\underset{\sim}{F})^2-36(\underset{\sim}{F})(\underset{\sim}{F}^2)-12(\underset{\sim}{F}^2)p-[8(\underset{\sim}{F})(\underset{\sim}{F}^2)$$

$$+\ 2(\underset{\sim}{F})^2](\underset{\sim}{\Omega})+[4(\underset{\sim}{F})p-20(\underset{\sim}{F})^2-24(\underset{\sim}{F})(\underset{\sim}{F}^2)-8(\underset{\sim}{F}^2)p](\underset{\sim}{\Omega}\underset{\sim}{F})$$

$$-\ 16[(\underset{\sim}{F})p+(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}^2)+[(\underset{\sim}{F})p+(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F})^2+(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F})^2].(p+1)$$

$$+\ [24(\underset{\sim}{F})-16(\underset{\sim}{F}^2)-15(\underset{\sim}{F})^3-32(\underset{\sim}{F}^2)(\underset{\sim}{F})^2](\underset{\sim}{\Omega})+[12p-8(\underset{\sim}{F})^3-32(\underset{\sim}{F}^3)$$

$$-\ 24(\underset{\sim}{F})-84(\underset{\sim}{F}^2)-26(\underset{\sim}{F})^2-16(\underset{\sim}{F})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F})-[16p+200(\underset{\sim}{F})$$

$$+\ 32(\underset{\sim}{F}^2)+24(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}^2)+[p+10(\underset{\sim}{F})+(\underset{\sim}{F}^2)+(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F})^2$$

$$-\ 32(\underset{\sim}{F})-88(\underset{\sim}{F}^2)-12(\underset{\sim}{F})^3-12(\underset{\sim}{F})(\underset{\sim}{F}^2)-48(\underset{\sim}{F}^3)-32(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}')$$

$$-\ 64(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}^3)+(\underset{\sim}{\Omega}'\underset{\sim}{\Omega})(\underset{\sim}{F})^2+(\underset{\sim}{\Omega}\underset{\sim}{F}\underset{\sim}{\Omega}')(\underset{\sim}{F})^2+(\underset{\sim}{\Omega}'\underset{\sim}{F}\underset{\sim}{\Omega})(\underset{\sim}{F})^2$$

$$+ (\underset{\sim}{\Omega}'\underset{\sim}{F}^2\underset{\sim}{\Omega})(\underset{\sim}{F})^2+4(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F})+4(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}^2)(\underset{\sim}{F})\}/(32),$$

$$A_{23} = -\{[[(3/2)(\underset{\sim}{F})^2p+(3/2)(\underset{\sim}{F})^3](\underset{\sim}{\Omega}\underset{\sim}{F})+(1/2)(\underset{\sim}{\Omega})(\underset{\sim}{F})^3].(p+1)$$

$$+ [3(\underset{\sim}{F})p+12(\underset{\sim}{F})^2+3(\underset{\sim}{F}^2)(\underset{\sim}{F})+(1/2)(\underset{\sim}{F})^3](\underset{\sim}{\Omega}\underset{\sim}{F})+[3(\underset{\sim}{F})^2+(1/2)$$

$$(\underset{\sim}{F})^4](\underset{\sim}{\Omega})+3(\underset{\sim}{F})^2(\underset{\sim}{\Omega}\underset{\sim}{F}^2)\}/(12),$$ and the rest available in [76].

As an immediate result $h(\underset{\sim}{S}_2)$ can be expressed in the following manner

$$h(\underset{\sim}{S}_2) = \theta-[\frac{1}{4n}\sum_{J=0}^{4} a_J(m,p)G_{mp+2J}(\theta,\omega^2)$$

$$+\frac{1}{n}\sum_{J=0}^{6} A_J G_{mp+2J}(\theta,\omega^2)][G'(\theta)]^{-1}+ O(n^{-2}). \qquad (48)$$

Recall that θ is the appropriate percentile of the linear function of a non-central chi-square variable of the form $Y = \sum_{j=1}^{p} \lambda_j x_j^2(m,\omega^2)$, the λ_j's are the characteristic roots of $\underset{\sim}{A}\underset{\sim}{B}^{-1}$ and $G(\theta)$ is the c.d.f. of Y in terms of the percentile. Finally, we can state the following theorem:

Theorem 5. Let $\underset{\sim}{Z} = (\underset{\sim}{z}_1,\ldots,\underset{\sim}{z}_m)$ be a pxm random matrix of independently distributed column vectors,where $\underset{\sim}{z}_i$ has the density $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_1=\underset{\sim}{B}^{-1})$, and $n\underset{\sim}{S}_2$ distributed central Wishart $W(n,p,\underset{\sim}{\Sigma}_2 = \underset{\sim}{A}^{-1})$. Under the assumption that $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I}+\underset{\sim}{F}$ and $|Ch_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, an asymptotic expansion for the percentile of T is given by Eq (48).

The following are special cases of Eq (48).

Case 1. When terms involving $f_{ij}f_{kl}$ are negligible, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$, terms like $(\underset{\sim}{F})^2$, $(\underset{\sim}{F}^2)$ and $(\underset{\sim}{\Omega})(\underset{\sim}{F}^3)$ can be dropped.

Consequently, A_6 disappears and A_0, A_1 up to A_5 are reduced drastically and, finally, Eq (48) agrees with the result of Chattopadhyay [11] to the indicated order.

Case 2. Under the equality of the two covariance matrices, the deviation matrix is zero. Putting $(\underset{\sim}{F}) = 0$ in Eq. (48), we get Eq (6.4) of Siotani [13].

4. AN ASYMPTOTIC EXPANSION FOR THE C.D.F. OF $T = \operatorname{Tr} \underset{\sim}{S}_2^{-1} \underset{\sim}{Z}\underset{\sim}{Z}'$

In the following an asymptotic expansion for the c.d.f. of T to $O(n^{-1})$ is derived by using the method described earlier. Also, it is possible to write

$$\Pr\{\operatorname{Tr} \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = \int_R \Pr\{\operatorname{Tr} \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta | \underset{\sim}{S}_2\}\Pr\{d\underset{\sim}{S}_2\}$$

$$- \Theta \Pr\{\operatorname{Tr} \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$$

where Θ is given by Eq (39). It follows that

$$\Pr\{\operatorname{Tr} \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = G(\theta) - \frac{1}{n}[h_1(\underset{\sim}{A}^{-1})]G'(\theta) + O(n^{-2})$$

Under the assumptions of Theorem 5, we get

$$\Pr\{\operatorname{Tr} \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = G(\theta) + \frac{1}{4n}\sum_{J=0}^{4} a_j(m,p)G_{mp+2j}(\theta,\omega^2)$$

$$+ \frac{1}{n}\sum_{J=0}^{6} A_J G_{mp+2j}(\theta,\omega^2) + O(n^{-2}), \qquad (49)$$

where the a_j's and A_J's are presented earlier and $G(\theta)$ and $G_{mp}(\theta,\omega^2)$ are defined earlier. Then it follows:

Theorem 6. Under the assumptions of the previous theorem, an asymptotic expansion for the c.d.f. of T is given by Eq (49).

Similarly we can get the two special cases, as we pointed out in the previous pages.

5. NUMERICAL RESULTS

The expansion given by Eq (49) has been used here to compute the powers of the test when the departure from the null hypothesis occurs. The following table shows these powers. For this tabulation, the upper five percent points were taken from Pillai and Jayachandran [16].

TABLE II

POWERS OF T TEST UNDER VIOLATIONS FOR $p=2$, $m=3$, $\alpha=0.05$ WHEN THE DEVIATION MATRIX HAS EQUAL DEVIATION PARAMETERS

ω_1	ω_2	($\underset{\sim}{F}$)	Up to the order	n = 83
		.00001	0(1)	.0379288
			$0(n^{-1})$	.049277 (.049273)
0	.00001	.00015	0(1)	.037942
			$0(n^{-1})$	.049296 (.049344)
		.005	0(1)	.038399
			$0(n^{-1})$	.049433 (.04977)
		.00001	0(1)	.03793
			$0(n^{-1})$	.049279 (.049276)
0	.0001	.00015	0(1)	.037945
			$0(n^{-1})$	.049300 (.049347)
		.005	0(1)	.038403
			$0(n^{-1})$	.049636 (.05182)
		.00001	0(1)	.038085
			$0(n^{-1})$	.04944 (.049455)
0	.005	.00015	0(1)	.038098
			$0(n^{-1})$	.049437 (.049475)
		.005	0(1)	.038557
			$0(n^{-1})$	.0499188 (.05200)

The figures in ( ) are computed using Chattopadhyay expansion [11].

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